Modelling an Aquaponic Ecosystem Using Ordinary Differential Equations

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Abstract Aquaponic agriculture is a sustainable system which uses interdependent processes and has been growing in popularity. However, relatively little mathematical and other academic research has been conducted in the practice. In this paper, we develop a system of ordinary differential equations to model the population and concentration dynamics of the environment. Our model has an asymptotically stable non-trivial equilibrium, representing the inherent symbiotic relationship of the variables. Values of the nine parameters in the system are estimated from the research literature. We provide simulated results simulated results to illustrate the nature of solutions to the system, and we present and discuss a sensitivity analysis.

1 Introduction

Aquaponics is a closed-loop agricultural system which uses a symbiotic relationship between aquatic organisms and aquatic macrophytes. The system recirculates water through an aquaculture environment (fish in a designated body of water) and a hydroponic structure (aquatic plants in soilless water) to create a sustainable environment which fully conserves water and nutrients. The key motivation behind aquaponics is using waste produced by fish in the system as a nutrient source for the plants. This process not only allows the fish waste to act as a natural fertilizer for the plants but also inherently cleans the water to be returned to the fish [6]. Thus the environment is in a natural state of stability, which we seek to model using a system of differential equations. This paper discusses background theory in aquaponics, develops a compatible aquaponic model in the form of a system of differential equations, discusses equilibria of the model and their stability, presents specific solution simulations for parameter values derived from the research literature, performs a sensitivity analysis, and suggests future research direction.

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2 Background: Research of Aquaponic Environments

Aquaponic agriculture systems have been growing in popularity due to their robust economic and ecological benefits. Both hydroponics and aquaculture have proven to have a detrimental effect on the environment. Some studies have suggested that aquaponics has a 75 % smaller carbon footprint when compared to traditional farming methods [7] and is able to satisfy human demand long-term [3]. Importantly, aquaponics achieves all of these benefits while maintaining economic viability for farmers. The lower cost of resources and lower spatial requirements has led to suggestions that aquaponic agriculture may be a viable solution in densely populated urban regions such as Pakistan and India [3].

The basic aquaponic system consists of two main components: one for fish and one for plants. The system relies heavily on food input as many studies have demonstrated that factors like protein content of food and feeding frequency have the largest effect on the efficiency of the system. These factors highly contribute to fish growth and are also directly related to the amount of fish waste in the environment [6].

Organic nitrogen in fish waste naturally converts to ammonia through biological degradation [4]. Ammonia is highly toxic to fish, and is an inefficient nutrient source for plants [6]. In order for ammonia to be used as fertilizer for the plants, it must go through a natural microbial process called the Nitrogen Cycle, which causes it to convert into nitrate. Nitrate is a nutrient rich food source for plants [4] and studies have shown that plants' uptake efficiency of nitrates ranges from 86 % to 98 %. Whatever concentration of nitrate is left in the water is not harmful to the fish, so the water can be recirculated for fish use [6].

3 Development of the Model

The overarching goal of the model is to capture the symbiotic relationship between the fish and the macrophytes. However, as evidenced by background research on aquaponic agriculture, the aerobic microbial process which converts ammonia to nitrate needs to be considered. The assumpions made are as follows:

i. The aquaponic ecosystem is a closed environment.
ii. The fish population increases at some natural survival rate $(\frac{\text{Births}}{\text{Deaths}})$, hindered by a carrying capacity due to the limited tank space.
iii. There is additional fish decay due to increased ammonia presence in the water until it reaches a critical ammonia level where no fish survive. This can be reasonably modelled using a linear constant of $(\frac{\text{Ammonia}}{\text{Toxic Ammonia Level}})$.
iv. Ammonia is present in the system exclusively due to fish waste and hence grows at a rate proportional to the fish population. It decays due to its conversion to nitrate.

v. Nitrate grows at a rate proporational to the level of ammonia, and decays due to plant uptake.
vi. Plants grow at a constant rate hindered by a carrying capacity indicative of the limited surface area of the system.
vii. Modelling the concentrations of ammonia and nitrate in the system will capture any other relationships between other variables in the nitrogen cycle.
viii. The system is well mixed so the nitrogen cycle occurs naturally and plants have even access to nitrate.

The proposed model is below, with variables F, A, N, and P representing the population of fish, ammonia (in mg), nitrate (in mg) and population of plants respectively:

$$\dot{F} = a_1\left(1 - \frac{F}{K_F}\right)FP - \frac{A}{K_A}F \tag{1}$$

$$\dot{A} = a_2F - a_3A \tag{2}$$

$$\dot{N} = a_4A - a_5NP \tag{3}$$

$$\dot{P} = a_6\left(1 - \frac{P}{K_P}\right)PN \tag{4}$$

where $a_i \geq 0\ \forall i$ are growth and decay rates, and $K_F, K_A, K_P > 0$ are the carrying capacities of fish, ammonia, and plants respectively.

Equation (1) models the evolution of the fish population, using assumptions ii and iii. Equation (2) models the evolution of the ammonia concentration in the system using assumption iv. Equation (3) captures the growth rate of nitrate concentration as it relates to the conversion from ammonia and the decay rate due to plant uptake using assumption v. The final equation (4) captures the growth rate of the plants using assumption vi.

4 Analyzing the Equilibria

Our aquaponic environment model (1), (2), (3) and (4) consists of four equations with nine unknown parameters. The Jacobian matrix of this model is as follows:

$$Df(F,A,N,P) = \begin{bmatrix} -\frac{a_1FP}{K_F} + a_1\left(1-\frac{F}{K_F}\right)P - \frac{A}{K_A} & -\frac{F}{K_A} & 0 & a_1\left(1-\frac{F}{K_F}\right)F \\ a_2 & -a_3 & 0 & 0 \\ 0 & a_4 & -a_5P & -a_5N \\ 0 & 0 & a_6P\left(1-\frac{P}{K_P}\right) & a_6N\left(1-\frac{P}{K_P}\right) - \frac{a_6NP}{K_P} \end{bmatrix} \tag{5}$$

The system (1) (2), (3) and (4) has three equilibria:

$$\{F = 0, A = 0, N = 0, P = P\} \tag{6}$$

$$\{F = 0, A = 0, N = N, P = 0\} \tag{7}$$

$$\Big\{F = \frac{a_1 a_3 K_F K_A K_P}{a_1 a_3 K_P K_A + a_2 K_F}, A = \frac{a_1 a_2 K_F K_A K_P}{a_1 a_3 K_A K_P + a_2 K_F},$$
$$N = \frac{a_1 a_2 a_4 K_F K_A}{a_5 \left(a_1 a_3 K_A K_P + a_2 K_F\right)}, P = K_P\Big\} \tag{8}$$

of which the third, in equation (8), since it represents all variables surviving in the environment. We focus on it in the next section.

4.1 Equilibrium (8): Coexistence

We analyzed the stability of the nontrivial equilibrium in equation (8) using the research literature to provide estimates for the nine parameters: $a_1 = 0.0124$, $a_2 = 0.1$, $a_3 = 0.94$, $a_4 = 3.6$, $a_5 = 0.92$, $a_6 = 0.056$, $K_F = 250$, $K_P = 300$ and $K_A = 20$ [1–4, 6]. The values selected for our carrying capacities were based on an arbitrary initial tank size of 10L, however, this can easily be scaled up or down to accomodate systems of various sizes. Other initial values were $\{F(0) = 10, A(0) = 0, N(0) = 0, P(0) = 0.5\}$, where $P(0) = 0.5$ was used to represent plants which had not yet reached maturity in the system. Substituting the estimated parameter values in the Jacobian gives:

$$A_* = \begin{bmatrix} -2.77 & -9.31 & 0 & 0.59 \\ 0.1 & -0.98 & 0 & 0 \\ 0 & 3.6 & -276 & -0.228 \\ 0 & 0 & 0 & -0.014 \end{bmatrix} \tag{9}$$

The eigenvalues of A_* in this particular simulation are:

$$\lambda_{31*} = -1.84 + 0.332i \tag{10}$$

$$\lambda_{32*} = -0.014 \tag{11}$$

$$\lambda_{33*} = -276 \tag{12}$$

$$\lambda_{34*} = -1.84 - 0.332i \tag{13}$$

Notably, the real parts of (10) and (13) are negative, and values (11) and (12) are negative, thus an asymptotically stable equilibrium is achieved in this case.

Because (10) and (13) are complex, some spiralling behaviour is present in the system.

Note that λ_{32*} is very close to 0, so the stability status of this equilibrium point may be sensitive to the parameter values chosen. Some experimentation was done within ranges of realistic values for the estimated parameters. In every case, all eigenvalues have negative real parts suggesting with some generality that the equilibrium case where all four variables are present in the system is asymptotically stable. We discuss sensitivity in general in the next section.

Substituting these values into the non-trivial equilibrium and subsequent solution given in the previous section provides the following equilibrium points:

$$\{F = 186.18, A = 19, N = 0.25, P = 300\} \tag{14}$$

The real-world context of these equilibrium points is exciting; it suggests that fish grow to a point well bounded by their carrying capacity, ($K_F = 250$). Ammonia tends towards an amount just below the threshold to which all the fish would die ($K_A = 20$). This suggests that the system is stabilizing in a way which approaches the maximum level of food production for the plants. Nitrate levels appear to be quite low, which is likely due to the superior nutritional uptake of the plants which has been noted in the literature. The plants themselves reach equilibrium at their carrying capacity ($K_P = 300$). This is exactly as anticipated based on the model structure.

Figure 1 shows two plots of the modelled variables over time given the parameter assumptions. As evidenced here, all the variables are approaching their above equilibrium values.The fish and plant follow a logistic trend tending to their carrying capacities. Ammonia also seems to be following this trend, which is due to its

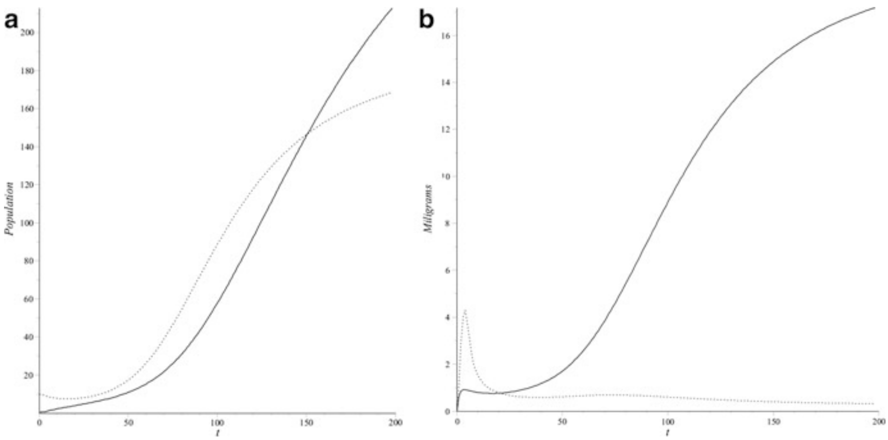

Fig. 1 The population and concentration dynamics of the model over time (**a**) The *solid line* represents the fish population and the *dotted line* represents the plant population (**b**) The *solid line* represents the nitrate population and the *dotted line* represents the ammonia population

growth dependence on fish. Nitrate also shows an interesting trend; as the system is being established, nitrate experiences a brief spike. However, once the plant growth reaches a level which requires significant nitrate sustenance, this spike reverses and the nitrate concentration levels off. These curves appear to loosely represent expected trends based on the aquaponics literature, suggesting the model is reasonably capturing aquaponic behaviour.

5 Sensitivity Analysis

A sensitivity analysis for the parameters of this model was performed with the goals of identifying any unexpected behaviour, guiding any data collection efforts, and most importantly, to give an indication of the importance of accurately estimating the parameter values.

A software toolbox in Matlab was used to perform sensitivity analysis of biological models. SensSB is freely available for academic purposes, and combines a variety of local and global sensitivity methods, both using relative and absolute measures to achieve many of these goals [8, 9].

While locally analysing the sensitivity of parameters is a useful exercise, it has an inherent reliance on the initial numerical estimation of the parameter. Global methods, which test the effect of a parameter while other parameters are varied simultaneously, help avoid this stipulation [9]. Since the initial estimation of the variables from the literature are considered weak points of the model, global sensitivity measures were analysed.

SensSB analyses global sensitivity through three main methods, all of which are discussed in the software documentation (see [9]). In this study, Derivative Based Global Sensitivity Measures (DBGSM), which were introduced in 2009 [5], were selected to optimize accuracy with minimal loss of computational efficiency [9]. DBGSM uses Monte Carlo sampling methods to average local derivatives to a measure $\bar{M}_{ij}$ which averages sensitivity measures S_{ij} over the parameter space.

$$S_{ij} = \frac{\%\text{ change in Parameters}}{\%\text{ change in Variables}} \tag{15}$$

$$\bar{M}_{ij} = \int_{H^n p} S_{ij} \; dp \tag{16}$$

As seen in Fig. 2, parameter 3 (a_3), or the conversion rate of ammonia to nitrate, has high global absolute sensitivity. In fact, according to the output from SensSB, it accounts for 73.26 % of the total sensitivity in the model. Notably, the relative sensitivity by variable, which shows how sensitive each variable is to the parameters in the model, shows high relative sensitivity for parameters a_2, a_3, a_6, and K_F. In the model, these parameters represent the growth rate of ammonia from fish waste, the conversion rate of ammonia to nitrate, the growth rate of plants and the carrying

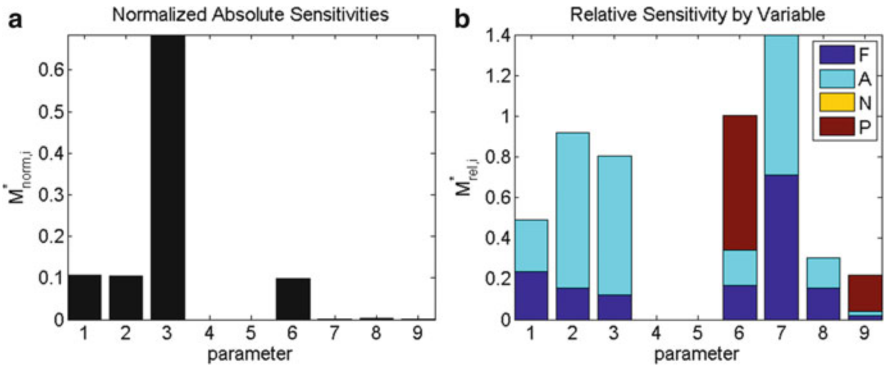

Fig. 2 (**a**) shows the global absolute sensitivity of each of the estimated parameters while (**b**) shows the global relative sensitivity by variable. Note parameters 1–6 are a_1 through a_6, parameter 7 is the carrying capacity K_F, parameter 8 is the carrying capacity K_A, and parameter 9 is the carrying capacity K_P

capacity of fish. This suggests that some more care should be taken in estimating values for these parameters, particularly a_2 and a_3, which do not have reliable values in the literature.

Using values from the previous simulation, both a_2 and a_3 were varied (individually and consecutively) between values of 0 and 100 without loss of stability in the coexistence equilibrium. However, exploring data from an established environment may help clear potential biases associated with this parameter estimation.

6 Future Research Direction

Future models of aquaponic systems should incorporate additional complexity. In particular, removing the assumption that the environment is closed to allow additional harvesting cycle terms for fish and plants would add realism and require additional analysis. Here, the term harvesting refers to the biological process of removing some fish and/or plants, creating a discontinuity in the solution.

As well, there is some indication in the literature that the assumption that the fish population decays linearly with increases in ammonia concentraion is invalid. Instead, it is expected that the fish population would not be affected by the presence of ammonia in the system until the ammonia begins to approach a critical level [1]. Instead, a non-linear polynomial relationship could be used, and may be a better approximation of the real world relationship.

Building from the sensitivity analysis, real data sets could be used to solve the parameter estimation inverse problem to see how well this model, and more complex variations, can accommodate real-world data as an approximation solution.

7 Conclusion

The developed four equation model appears to reasonably simulate biological behaviour. Simulated solutions lead to a few conclusions which may or may not hold in the general case. The most important of these is that the model admits an asymptotically stable equilibrium where fish, ammonia, nitrate, and plants interdependently coexist in the environment. An exploration of parameter space suggests that the stability of such an equilibrium point is robust to changes in the parameters.

Aquaponics shows considerable promise as an ecological agriculture solution. It is the hope that this research can be used to further test the ability of aquaponic solutions as viable practices in both developed and undeveloped countries to aid in concerns over the sustainability of current food practices.

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